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A few more RBIBDs with $k = 5$ and $\lambda = 1$*

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Abstract

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It has been shown that there exists a $(v, 5, 1)$-RBIBD for any positive integer $v \equiv 5 \pmod{20}$ with 147 possible exceptions. We show that such designs exist for 34 of these values.

1. Introduction

A *balanced incomplete block design* (BIBD) with parameters (v, k, λ) is a pair $(X, \mathcal{A})$ where X is a v-set and $\mathcal{A}$ is a family of k-subsets (where $2 < k < v$) called blocks, in which $\mathcal{A}$ has the property that every pair of distinct points of X occurs in precisely λ of its members. A *parallel class* of blocks of a design $(X, \mathcal{A})$ is a subclass $\mathcal{A}_1 \subset \mathcal{A}$ such that each point $x \in X$ is contained in exactly one block of $\mathcal{A}_1$, i.e., $\mathcal{A}_1$ is a partition of X. A BIBD $(X, \mathcal{A})$ is *resolvable* if $\mathcal{A}$ can be partitioned into parallel classes.

It is well known that:

(i) $\lambda(v-1) \equiv 0 \pmod{k-1}$,

(ii) $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$, and,

(iii) $v \equiv 0 \pmod{k}$,

are necessary conditions for the existence of a resolvable BIBD, (v, k, λ)-RBIBD. For $k = 5$ and $\lambda = 1$, these reduce to the condition that v be congruent to $5 \pmod{20}$.

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It is conjectured by Ray-Chaudhuri and Wilson [7] that a $(v, 5, 1)$-RBIBD exists if and only if $v \equiv 5 \pmod{20}$. In two previous papers [3, 9], it has been shown that for any positive integer $v \equiv 5 \pmod{20}$ there exists a $(v, 5, 1)$-RBIBD with at most 147 possible exceptions of v, in which 23085 is the largest.

It is our purpose here to reduce this number of possible exceptions to 113, in which 7845 becomes the largest. We also present a new construction to obtain an RBIBD from an RBIBD and a BIBD.

Since a TD(6, 20) exists from [8], applying some known constructions we obtain some new $(v, 5, 1)$-RBIBD.

Let $(X, \mathscr{A})$ be a $(v, k, 1)$-RBIBD having parallel classes $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r$. Let $(Y, \mathscr{B})$ be a $(u, k, 1)$-RBIBD having parallel classes $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_t$. If $X \supset Y$ and $\mathscr{A}_i \supset \mathscr{B}_i$, $1 \leq i \leq t$, we say that the first design contains the second as a subdesign. The following is shown in [6].

Theorem 1.1. *If there exists a* $(v, k, 1)$-RBIBD, *a* $((k-1)m + w, k, 1)$-RBIBD *with a subdesign* $(w, k, 1)$-RBIBD *(or* $w = 1$*), and a* $\mathrm{TD}(k+1, m)$, *then there exists an* $(m(v-1) + w, k, 1)$-RBIBD.

Lemma 1.2. *There exists a* (1285, 5, 1)-RBIBD.

Proof. Apply Theorem 1.1 with $v = 65$, $k = 5$, $m = 20$ and $w = 5$. Since a $(v, 5, 1)$-RBIBD exists for $v = 65$ and 85, we obtain a $(u, 5, 1)$-RBIBD, where $u = 20(65-1) + 5 = 1285$. □

Lemma 1.3. *There exists a* (1345, 5, 1)-RBIBD.

Proof. Apply Theorem 1.1 with $v = 65$, $k = 5$, $m = 21$ and $w = 1$. Since a TD(6, 21) exists (see, for example, [1]), we obtain a $(u, 5, 1)$-RBIBD, where $u = 21(65-1) + 1 = 1345$. □

The next construction involves the PBD-closed set R_k^*, where

$$R_k^* = \{r: a\ (r(k-1)+1,\ k,\ 1)\text{-RBIBD exists}\}.$$

We employ the singular indirect product for PBDs.

Lemma 1.4. $171 \in R_5^*$.

Proof. From Mullin [5, Lemma 3.13] we have a $\mathrm{TD}(6, 28) - \mathrm{TD}(6, 3)$. Add a set Z of three new points to the incomplete TD and construct a $(31, 6, 1)$-BIBD on the set $G \cup Z$ for each group G such that the three points in both G and the missing TD(6, 3) form a block together with the points of Z. We then obtain a $(171, \{6, 21\}, 1)$-PBD. Since $6, 21 \in R_5^*$, we know that $171 \in R_5^*$. □

We shall need some incomplete TDs which can be obtained from the following lemma.

Lemma 1.5. *Suppose that a, t, and m are integers satisfying $0 \leq t \leq m$ and $0 \leq a \leq m$. If there exist a* TD$(8, m)$ *and a* TD$(6, t)$, *then there exists a* TD$(6, 7m + t + a) -$ TD$(6, a)$.

Proof. A corollary of Theorem 1.1 in [2]. □

Corollary 1.6. *There exist* TD(6, 166) − TD(6, 16), TD(6, 436) − TD(6, 16) *and* TD(6, 73) − TD(6, 3).

Proof. Apply Lemma 1.5 with $m = 19$, $t = 17$ and $a = 16$ to obtain the first incomplete TD. Take $m = 59$, $t = 7$ and $a = 16$ to obtain the second design. The third one can be obtained by taking $m = 9$, $t = 7$ and $a = 3$. The required TD$(8, m)$ and TD$(6, t)$ come from [1, Table H]. □

Lemma 1.7. $1001, 2621 \in R_5^*$.

Proof. Add five new points to TD(6, 166) − TD(6, 16) and break each group (and the new points) with a (171, {6, 21}, 1)-PBD from Lemma 1.4. We obtain a (1001, {6, 101}, 1)-PBD. Since $101 \in R_5^*$ (see [3]), it follows that $1001 \in R_5^*$. Adding five new points to TD(6, 436) − TD(6, 16) and breaking the groups with a (441, {6, 21}, 1)-PBD, we obtain a (2621, {6, 101}, 1)-PBD and $2621 \in R_5^*$. Here the required (441, {6, 21}, 1)-PBD comes from TD(6, 73) − TD(6, 3) by adding 3 new points and using a (76, 6, 1)-BIBD to break the groups. □

Lemma 1.8. $676 \in R_5^*$.

Proof. Adding a new point to a TD(6, 20) we obtain a (121, {6, 21}, 1)-PBD. Apply Lemma 1.5 with $m = 13$, $t = 9$ and $a = 11$ to obtain a TD(6, 111) − TD(6, 11). Add ten new points and break the groups with the PBD. We obtain a (676, {6, 21, 76}, 1)-PBD. Since $76 \in R_5^*$, then $676 \in R_5^*$. □

We need another corollary of [2, Theorem 1.2].

Lemma 1.9. *If* TD$(6 + w, t)$, TD$(6, m)$, TD$(6, m + 1)$ *and* TD$(6, m + w)$ *all exist, then there exist* TD$(6, mt + w) -$ TD$(6, t)$ *and* TD$(6, mt + w) -$ TD$(6, m + w)$.

Lemma 1.10. $701 \in R_5^*$.

Proof. Take $t = 16$, $m = 7$ and $w = 4$ in Lemma 1.9 to obtain a TD(6, 116) − TD(6, 16). Adding five new points and using a (121, {6, 21}, 1)-PBD to break the groups we obtain a (701, {6, 21, 101}, 1)-PBD. Then, $701 \in R_5^*$. □

Let $B(K)=\{v: a\ (v, K, 1)\text{-PBD exists}\}$. Denote by $(v, K\cup\{q^*\}, 1)$-PBD a PBD which has exactly one block of size q and other block sizes in K. (Note that if $q\in K$, then a $(v, K\cup\{q^*\}, 1)$-PBD is a $(v, K, 1)$-PBD with at least one block of size q, and if $q=1$, then a $(v, K\cup\{q^*\}, 1)$-PBD is simply a $(v, K, 1)$-PBD.) We use the notation $v\in B(K\cup\{q^*\})$ to indicate the existence of a $(v, K\cup\{q^*\}, 1)$-PBD.

Lemma 1.11. *Suppose*:
(1) *a* $\mathrm{TD}(22, t)$ *exists*;
(2) $5t+q\in B(R_5^*\cup\{q^*\})$, $q\leqslant 101$;
(3) $20u+5v+q\in B(R_5^*\cup\{q^*\})$, $u+v=t$.
Then, $r=100t+20u+5v+101\in R_5^*$.

Proof. Give weight 5 to each point of a $\mathrm{TD}(22, t)$ except $t-m$ points in the first group which are given weight zero and u points in the second group which are given weight 20 each. Add q new points to the design such that $5m+q=101$. Using (2) and (3) to break the groups we obtain $r\in R_5^*$ if we have appropriate input GDDs. The input GDDs needed are given below. Since 106, $111\in B(6)$ (see [5, Lemma 3.2]), we can delete one point from a $(106, 6, 1)$-BIBD and a $(111, 6, 1)$-BIBD to get 6-GDDs of type 5^{21} and 5^{22}. Delete one point from a $\mathrm{TD}(6, 20)$ and add a new point to the design. We obtain a $\{6, 21\}$-GDD of type $5^{20}20^1$. Delete one point from a $\mathrm{TD}(6, 21)$ to obtain a $\{6, 21\}$-GDD of type $5^{21}20^1$. Then, $r\in B(\{6, 21, 101\}\cup R_5^*)\subset R_5^*$. □

Lemma 1.12. $\{2771, 2921, 3041, 3071, 3161, 3221, 3491, 3521, 4346, 4361, 4391, 4421, 5621\}\subset R_5^*$.

Proof. Apply Lemma 1.11 with the parameters in Table 1. Since t is prime power in Table 1, (1) is satisfied. For (2) and (3), 151, 241, $826\in B(6)$ come from [5,

Table 1

r	t	u	v	q	$5t+q$	$20u+5v+q$
2771	25	3	22	1	126	171 (Lemma 1.4)
2921	25	13	12	1	126	321
3041	25	21	4	1	126	441
3071	25	23	2	1	126	471
3161	27	15	12	1	136	361
3221	27	19	8	1	136	421
3491	29	23	6	6	$151\in B(6)$	$496\in B(R_5^*\cup\{6^*\})$
3521	29	25	4	6	$151\in B(6)$	$526\in B(R_5^*\cup\{6^*\})$
4346	37	24	13	1	186	546
4361	37	25	12	1	186	561
4391	37	27	10	1	186	591
4421	37	29	8	1	186	621
5621	47	39	8	6	$241\in B(6)$	$826\in B(6)$

Lemma 3.2]. $321 \in R_5^*$ comes from Lemma 1.2. The other numbers all belong to R_5^* from [9]. Add 16 new points to a TD(6, 80) and break the groups with a TD(6, 16). We obtain $496 \in B(\{6, 16\})$ and then $496 \in B(R_5^* \cup \{6^*\})$. Take $m = 11$, $t = 8$ and $a = 2$ in Lemma 1.5 to obtain a TD(6, 87) − TD(6, 2). Add four new points to the design and break the groups with a (91, 6, 1)-BIBD. We obtain a (526, $\{6, 16\}$, 1)-PBD and then $526 \in B(R_5^* \cup \{6^*\})$. The proof is complete. □

2. A new construction

In this section we present a new recursive construction to obtain an RBIBD from an RBIBD and a BIBD. Therefore, we can construct more RBIBDs with $k = 5$ and $\lambda = 1$.

A subset of blocks in a BIBD is called a *partial parallel class* if the subset consists of pairwise disjoint blocks.

Theorem 2.1. *Suppose $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are a $(u, k, 1)$-*RBIBD *and a $(v, k, 1)$-*BIBD, *respectively. Suppose $\mathscr{B}$ can be partitioned into s disjoint partial parallel classes $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_s$, where $s \leq (u + v - 2)/(k - 1)$. If there is an* RTD$(k, v)$, *then there exists a $(uv, k, 1)$-*RBIBD.

Proof. Let $u = (k-1)r_1 + 1$ and $v = (k-1)r_2 + 1$. Denote the r_1 parallel classes of $\mathscr{A}$ by $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{r_1}$. For any $A \in \mathscr{A}$ construct an RTD(k, v) on $A \times Y$ with groups $\{a\} \times Y$, $a \in A$, and blocks $\mathscr{D}^A$ where $\mathscr{D}^A$ is partitioned into parallel classes $\mathscr{D}_i^A$, $1 \leq i \leq v$. Let $\mathscr{D}^{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} \mathscr{D}^A$. Construct on $\{x\} \times Y$, a $(v, k, 1)$-BIBD $(\{x\} \times Y, \mathscr{B}^x)$ and let $\mathscr{B}^X = \bigcup_{x \in X} \mathscr{B}^x$. It is easy to see that $(X \times Y, \mathscr{D}^{\mathscr{A}} \cup \mathscr{B}^X)$ is a $(uv, k, 1)$-BIBD. To prove resolvability, let $\mathscr{D}_i^{\mathscr{A}_j} = \bigcup_{A \in \mathscr{A}_j} \mathscr{D}_i^A$ to obtain $r_1 v$ parallel classes $\mathscr{D}_i^{\mathscr{A}_j}$. Without loss of generality, we may assume

$$\mathscr{D}_1^{\mathscr{A}_j} = \{A \times \{y\} \mid A \in \mathscr{A}_j, y \in Y\}, \quad 1 \leq j \leq r_1.$$

We shall prove that $\mathscr{B}^X \cup (\bigcup_{1 \leq j \leq n} \mathscr{D}_1^{\mathscr{A}_j})$ can be partitioned into s parallel classes on $X \times Y$, where $n = s - r_2 \leq r_1$. Since

$$\mathscr{D}^A \Big\backslash \Big(\bigcup_{1 \leq j \leq n} \mathscr{D}_1^{\mathscr{A}_j} \Big)$$

contains $r_1 v - n$ parallel classes, $\mathscr{D}^{\mathscr{A}} \cup \mathscr{B}^X$ will consist of $r_1 v - n + s = r_1 v + r_2$ parallel classes and then form a (uv, k, l) − RBIBD.

Let $\mathscr{B}_i^x = \{\{x\} \times B \mid B \in \mathscr{B}_i\}$ and $\mathscr{B}_i^X = \bigcup_{x \in X} \mathscr{B}_i^x$. Suppose $S_i = Y \backslash (\bigcup_{B \in \mathscr{B}_i} B)$, $1 \leq i \leq s$. Then

$$\sum_{1 \leq i \leq s} |S_i| = vs - \sum_{B \in \mathscr{B}} |B| = vs - vr_2 = vn$$

and any point of Y appears in exactly n of the sets $S_1, \ldots, S_s$. Let

$$\mathscr{A}_j^y = \{A \times \{y\} \mid A \in \mathscr{A}_j\} \quad \text{and} \quad \mathscr{A}_j^S = \bigcup_{y \in S} \mathscr{A}_j^y.$$

So, $\mathscr{D}_1^{\mathscr{A}_j} = \mathscr{A}_j^Y$. It is not difficult to see that S_i can be partitioned into S_{ij}, $1 \leqslant j \leqslant n$, such that

$$\bigcup_{1 \leqslant i \leqslant s} \left(\bigcup_{1 \leqslant j \leqslant n} \mathscr{A}_j^{S_{ij}} \right) = \bigcup_{1 \leqslant j \leqslant n} \mathscr{D}_1^{\mathscr{A}_j}.$$

For example, we can easily choose S_{1j}, $1 \leqslant j \leqslant n$ and define S_{ij} recursively. Suppose S_{ij} ($1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant n$) are already chosen. Let $Y_{tj} = Y \setminus (\bigcup_{1 \leqslant i \leqslant t} S_{ij})$. We can define $S_{t+1,j}$ ($1 \leqslant j \leqslant n$) as follows:

$$S_{t+1,1} = Y_{t1} \cap S_{t+1}, \qquad S_{t+1,2} = Y_{t2} \cap (S_{t+1} \setminus S_{t+1,1}),$$

$$S_{t+1,3} = Y_{t3} \cap (S_{t+1} \setminus (S_{t+1,1} \cup S_{t+1,2})), \ldots,$$

$$S_{t+1,n} = Y_{tn} \cap \left(S_{t+1} \setminus \left(\bigcup_{1 \leqslant j \leqslant n-1} S_{t+1,j} \right) \right).$$

Let $E_i = \bigcup_{1 \leqslant j \leqslant n} \mathscr{A}_j^{S_{ij}}$. We then know that $E_i \cup \mathscr{B}_i^X$ is a parallel class on $X \times Y$ for $1 \leqslant i \leqslant s$. This completes the proof. □

Lemma 2.2. *There exists a* $(20t + 5, 5, 1)$-RBIBD *for* $t = 68$, 89, 131, 215, 278, 341 *and* 404.

Proof. It is obvious that the blocks in a $(21, 5, 1)$-BIBD can be partitioned into 21 partial parallel classes, each containing one block. Suppose there is a $(4r + 1, 5, 1)$-RBIBD. Then, Theorem 2.1 can be used to produce a $(21(4r + 1), 5, 1)$-RBIBD if $21 \leqslant r + 5$, i.e., $r \geqslant 16$. Since $r = 16$, 21, 31, 51, 66, 81, $96 \in R_5^*$ from [9], we obtain a $(20t + 5, 5, 1)$-RBIBD for $t = (21r + 4)/5$. Then, the conclusion follows. □

Lemma 2.3. *There exists a* $(45, 5, 1)$-BIBD *such that the blocks can be partitioned into* 17 *partial parallel classes.*

Proof. Using a primitive polynomial $f(x) = x^2 - x - 1$ on GF(3), we obtain GF(9) with primitive root x. By Hanani [4, Lemma 4.13] we have a $(45, 5, 1)$-BIBD with blocks generated under the additive group of GF(9) from the following base blocks:

$$B_0 = \{0_0, 0_1, 0_2, 0_3, 0_4\},$$

$$B_{ij} = \{0_j, x_{j+1}^i, x_{j+1}^{i+4}, x_{j+4}^{i+2}, x_{j+4}^{i+6}\}, \quad j \in Z_5, \ 0 \leqslant i \leqslant 1.$$

Denote

$$A_1 = \{0, x^2, x^6\}, \quad A_2 = \{x^1, x^3, x^4\}, \quad A_3 = \{x^5, x^0, x^7\},$$
$$H_1 = \{0, x^1, x^5\}, \quad H_2 = \{x^2, x^3, x^0\}, \quad H_3 = \{x^4, x^7, x^6\},$$
$$M_1 = \{x^5, x^3, x^4\}, \quad M_2 = \{0, x^0, x^7\}, \quad M_3 = \{x^1, x^2, x^6\}.$$

That is,

$$A_2 = A_1 + x^1, \quad A_3 = A_1 + x^5,$$
$$H_2 = H_1 + x^2, \quad H_3 = H_1 + x^4,$$
$$M_2 = M_1 + x^1, \quad M_3 = M_1 + x^5.$$

It is readily checked that the blocks can be partitioned into 17 partial parallel classes as follows:

$$\mathcal{B}_i = \{B_{10} + g,\ B_{11} + g \mid g \in A_i\}, \quad i = 1,2,3,$$
$$\mathcal{B}_{3+j} = \{B_{12} + g,\ B_{13} + g \mid g \in A_j\}, \quad j = 1,2,3,$$
$$\mathcal{B}_{6+s} = \{B_{14} + g,\ B_{01} + h_s,\ B_{03} + h_s \mid g \in A_s\}, \quad s = 1,2,3,$$

where $h_1 = 0,\ h_2 = x,\ h_3 = x^5$,

$$\mathcal{B}_{10} = \{B_{04} + g,\ B_{00} + g \mid g \in H_1\},$$
$$\mathcal{B}_{11} = \{B_{01} + g,\ B_{02} + g \mid g \in H_3\},$$
$$\mathcal{B}_{12} = \{B_{03} + g,\ B_{04} + g \mid g \in H_3\},$$
$$\mathcal{B}_{13} = \{B_{00} + g \mid g \in H_3\},$$
$$\mathcal{B}_{14} = \{B_{02} + g \mid g \in H_1\},$$
$$\mathcal{B}_{14+t} = \{B_{0i} + h_t,\ B_0 + m_t \mid i \in Z_5,\ m_t \in M_t\}, \quad t = 1, 2, 3,$$

where $h_1 = x^2,\ h_2 = x^3,\ h_3 = x^0$. □

Lemma 2.4. *There exists a* $(20t + 5, 5, 1)$-RBIBD *for* $t = 56$, 191 *and* 281.

Proof. Apply Theorem 2.1 with the $(45, 5, 1)$-BIBD from Lemma 2.3. Since the BIBD has altogether 17 partial parallel classes of blocks, a $(45(4r + 1), 5, 1)$-RBIBD exists if a $(4r + 1, 5, 1)$-RBIBD exists, where $17 \leqslant r + 11$, i.e., $r \geqslant 6$. Since $r = 6$, 21, $31 \in R_5^*$, we obtain a $(20t + 5, 5, 1)$-RBIBD for $t = 9r + 2$. Then, the conclusion follows. □

In order to delete the largest possible exception we need the following lemma (see [9, Theorem 2.20]).

Lemma 2.5. *Suppose*:
(1) *a* TD(17, t) *exists and* $5t + q \in B(R_5^* \cup \{q^*\})$;
(2) $15u + q \in B(R_5^* \cup \{q^*\})$, $0 \leq u \leq t$;
(3) $5v + q \in R_5^*$, $0 \leq v \leq t$.
Then, $75t + 15u + 5v + q \in R_5^*$.

Lemma 2.6. *There exists a* (23085, 5, 1)-RBIBD.

Proof. Apply Lemma 2.5 with $t = 73$, $q = 1$, $u = 1$ and $v = 56$. (1) and (2) are satisfied from [9, Theorem 1.1]. (3) is also satisfied from Lemma 2.4. Then, it follows that $75 \cdot 73 + 15 \cdot 1 + 5 \cdot 56 + 1 = 5771 \in R_5^*$. □

We also need a corollary of [2, Theorem 1.2].

Lemma 2.7. *If* TD($7 + w, t$), TD($6, m$), TD($6, m + 1$), TD($6, m + 2$) *and* TD($6, a$) *all exist for* $0 \leq a \leq t - 1$, *then there exists a* TD($6, mt + w + a$) − TD($6, m + w$).

Lemma 2.8. 2321, 2471, $2591 \in R_5^*$.

Proof. Since a $(20t + 5, 5, 1)$-RBIBD exists for $t = 56$, 68 from Lemmas 2.2 and 2.4, we know that 281, $341 \in R_5^*$. It has been proved in [9, Corollary 2.4] that $5r + 1 \in B(\{6, r^*\}) \subset R_5^*$ if $r \in R_5^*$. Since 91, $96 \in R_5^*$, we have $456 \in B(\{6, 91^*\})$ and $481 \in B(\{6, 96^*\})$. A TD(6, 422) − TD(6, 37) exists from Lemma 1.5 with $m = 55$, $t = 0$ and $a = 37$. Here only a TD($7, m$) is needed since $t = 0$. Adding 59 new points to the incomplete TD we obtain $2591 \in B(\{6, 96, 281\}) \subset R_5^*$. A TD(6, 403) − TD(6, 38) exists from Lemma 2.7 with $t = 53$, $m = 7$, $a = 1$ and $w = 31$. Adding 53 new points to the incomplete TD we obtain $2471 \in B(\{6, 91, 281\}) \subset R_5^*$. Finally, a TD(6, 77) − TD(6, 11) exists since $77 = 7 \cdot 11$. Giving each point weight 5 we have a TD(6, 385) − TD(6, 55) since a TD(6, 5) exists. Add 11 new points to the incomplete TD. Break the groups with a (396, {6, 66}, 1)-PBD which comes from a TD(6, 66). We obtain $2321 \in B(\{6, 66, 341\}) \subset R_5^*$. The proof is complete. □

3. Conclusion

It has been shown that a $(20u + 5, 5, 1)$-RBIBD exists for $u \in$ {34, 56, 64, 67, 68, 89, 131, 135, 140, 191, 200, 215, 278, 281, 341, 404, 464, 494, 518, 524, 554, 584, 608, 614, 632, 644, 698, 704, 869, 872, 878, 884, 1124, 1154}. Updating the result in [9] we obtain the following theorem.

Table 2

45	105	145	165	185	225	245	285	345	465
525	565	585	645	665	705	765	785	805	825
885	905	925	945	985	1005	1045	1065	1145	1165
1185	1245	1305	1385	1425	1485	1505	1545	1605	1665
1725	1845	1905	1965	2085	2145	2205	2265	2325	2385
2445	2505	2565	2685	2745	2865	2985	3045	3105	3165
3225	3345	3465	3525	3585	3645	3705	3765	3785	3885
3945	4065	4185	4245	4365	4425	4485	4545	4605	4665
4725	4785	4845	4905	4965	5025	5085	5145	5385	5445
5685	5745	5865	5925	5985	6045	6165	6225	6285	6345
6585	6645	6705	6945	7005	7065	7125	7185	7245	7365
7425	7485	7845							

Theorem 3.1. *For any* $v > 7845$, $v \equiv 5 \pmod{20}$, *there exists a* $(v, 5, 1)$-RBIBD. *Further, a* $(v, 5, 1)$-RBIBD *exists for any* $v \equiv 5 \pmod{20}$, $5 \leqslant v \leqslant 7845$, *with at most* 113 *possible exceptions, which are the values in Table* 2.

Note added in proof

Recently, Paul Schellenberg has obtained three new $(v, 5, 1)$-RBIBDs for $v = 805$, 905 and 1505 by using certain difference families. Malcolm Greig found *a* $(246, 6, 1)$-BIBD which implies the existence of *a* $(985, 5, 1)$-RBIBD. Therefore, there are 109 unsolved cases now.

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